Basic Finite Volume Concepts

in Computational Fluid Dynamics (CFD)

Computational Fluid Dynamics (CFD) has become an essential tool in modern engineering analysis and thermal management, leading to faster solutions and an increase in productivity. It is also a very rich field that has benefitted from decades of work to improve its accuracy and speed. Thus, today, engineers have at their disposal an array of different choices of CFD packages to solve their thermal problems. It is beneficial, however, if the user possesses a basic understanding of the underlying concepts, so that he or she can utilize the CFD software more effectively.

There are many different types of flows, such as the dimensionality of the flow and the nature of the turbulence and unsteadiness. There are also different techniques for solving flow problems, such as finite differences, finite volume and finite element. One of the most basic concepts is the application of finite volume for solving fluid flow and heat transfer. For simplicity we assume a 2-D problem.

The equations for the X, Y momentum and continuity for an incompressible flow assuming constant density are as follows:

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

$$\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3}$$

Where,

U = x component of velocity
V = v component of velocity
P = pressure
Re = Reynolds number

The computation domain is divided into cells. The cells can have equal sides, or they can be of different lengths. The staggered grid specifies the values of u and v at the faces of the control volume around the grid cells and the value of pressure at the grid point. Figure 1 shows a schematic of the staggered grid.

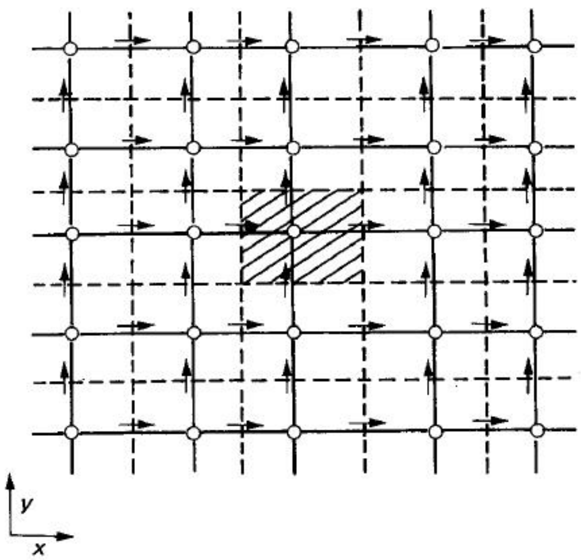

Figure 1. Schematic of a Staggered Grid [1]

Figure 1 shows a control volume around a grid cell shown as a circle, which is where the value of pressure is stored and the horizontal and vertical arrows are the x and y component of velocity. Figure 2 shows a control volume around the u and v velocity. The values of pressures $P_{i,j}$ and $P_{i+1,j}$ and $P_{j+1,i}$ are defined at the faces of the control volume and the value of u and v components are defined at the center of the control volumes.

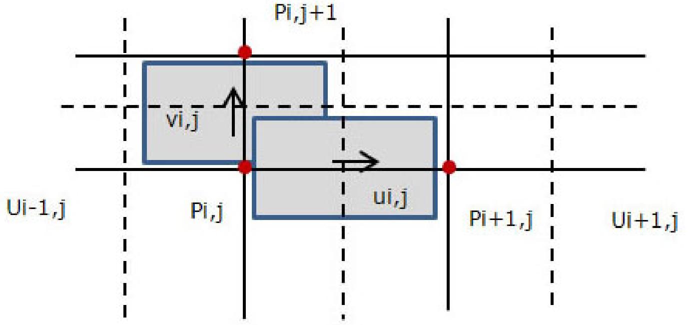

Figure 2. Control Volume for X and Y Momentum

Figure 3 shows the control volume for the continuity equation. Note that the values of pressure P is at the center of the control volume and values of u and v are at the faces of the control volume.

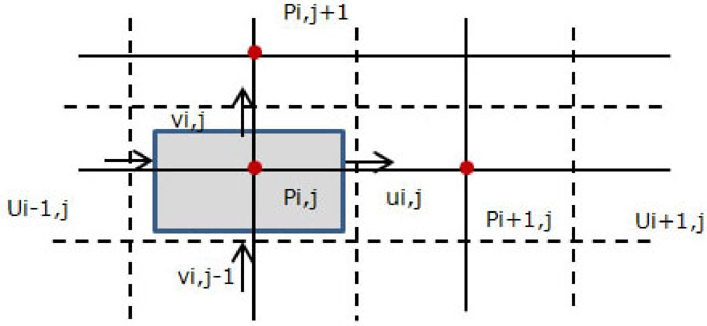

Figure 3. Control Volume for the Continuity Equation

Integration of the u momentum equation around this control volume will result in:

$$\frac{\Delta x\Delta y}{\Delta t}(u^{n+1}_{i,j}-u^{n}_{i,j})+\iint\frac{\partial}{\partial x}\left\{u^2-1/Re.\frac{\partial u}{\partial x}\right\}dxdy+\iint\frac{\partial}{\partial y}\left\{uv-1/Re.\frac{\partial u}{\partial y}\right\}dxdy+\iint\frac{\partial p}{\partial x}dxdy=0 \qquad (4)$$

Where the indices n and n+1 indicate the n and n+1 time step. Using Green's theorem and extensive algebraic manipulation, we get the following two sets of discretized equations:

$$\left(\frac{\Delta x\Delta y}{\Delta t}+a^{u}_{i,j}\right)u^{n+1}_{i,j}+\sum a^{u}_{nb}u^{n+1}_{nb}=-b^{u}-\Delta y(P^{n+1}_{i+1,j}-P^{n+1}_{i,j}) \qquad (5)$$

$$\left(\frac{\Delta x\Delta y}{\Delta t}+a^{v}_{i,j}\right)v^{n+1}_{i,j}+\sum a^{v}_{nb}v^{n+1}_{nb}=-b^{v}-\Delta x(P^{n+1}_{i,j+1}-P^{n+1}_{i,j}) \qquad (6)$$

The **a** and **b** letters are the constants calculated from the previous time step and **nb** refers to the neighboring points.

The above two equations and the continuity equations provide 3 equations and 3 unknowns (u, v and p) to be solved. However, there is not an obvious way to eliminate the pressure term from these equations. Attempts have been made to formulate the flow equations in terms of vorticity and stream function, which can readily result in the disappearance of the pressure term, but such techniques are harder to implement, especially for 3-D flows and the difficulty of specifying the stream function on interior surfaces.

One technique to circumvent the difficulty of calculating the pressure field is to make an educated guess at the values of the pressure and calculate the estimated values of velocity based on these estimated values. So, the above two equations can be written as:

$$\left(\frac{\Delta x\Delta y}{\Delta t}+a^{u}_{i,j}\right)u^{*}_{i,j}+\sum a^{u}_{nb}u^{*}_{nb}=-b^{u}-\Delta y(P_{i+1,j}-P_{i,j}) \qquad (7)$$

$$\left(\frac{\Delta x\Delta y}{\Delta t}+a^{v}_{i,j}\right)v^{*}_{i,j}+\sum a^{v}_{nb}v^{*}_{nb}=-b^{v}-\Delta x(P_{i,j+1}-P_{i,j}) \qquad (8)$$

The starred values of **u** and **v** are based on the guessed values of the pressure. These estimated values of velocity most likely do not satisfy the continuity equation. Let's assume the velocity values is composed of a guessed value and a correction.

$$u = u^* + u_c \tag{9}$$

$$v = v^* + v_c \tag{10}$$

Using these values, and subtracting equations 7 and 8 from equations 5 and 6, we arrive at the following equations for the correction in velocities in terms of differences in correction for the pressures.

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^u_{i,j}\right) u^c_{i,j} + \sum a^u_{nb} u^c_{nb} = -\Delta y(\delta P_{i+1,j} - \delta P_{i,j}) \tag{11}$$

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^v_{i,j}\right) v^c_{i,j} + \sum a^v_{nb} v^c_{nb} = -\Delta x(\delta P_{i,j+1} - \delta P_{i,j}) \tag{12}$$

The summation terms in the above equations can be dropped for computational reasoning. Its discussion is beyond the scope of this article. This results in the following two relations for the corrected values of velocity in terms of the correction of pressure:

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^u_{i,j}\right) u^c_{i,j} = -\Delta y(\delta P_{i+1,j} - \delta P_{i,j}) \tag{13}$$

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^v_{i,j}\right) v^c_{i,j} = -\Delta x(\delta P_{i,j+1} - \delta P_{i,j}) \tag{14}$$

Now, in order to link the pressure corrections to the velocities, we need to use the third equation which is the continuity equation. The continuity equation can be integrated around the control volume in Figure 3 which results in the following equation.

$$\frac{u_{i,j} - u_{i-1,j}}{\Delta x} + \frac{v_{i,j} - v_{i,j-1}}{\Delta y} = 0 \tag{15}$$

Substituting equations 9, 10, 13 and 14 into equation 15 yields a relation for the pressure correction at grid points.

$$\begin{aligned} a^p_{i,j}\delta p_{i,j} &= \sum a^p_{nb}\delta p_{nb} + b^p \\ b^p &= (u^*_{i,j} - u^*_{i-1,j})\Delta y + (v^*_{i,j} - v^*_{i,j-1})\Delta x \end{aligned} \tag{16}$$

The solution procedure is as follows:

1. Guess the values of the pressure
2. Find values of u* and v* from equations 7 and 8
3. Find values of δp from equation 16
4. u_c and v_c are calculated from equations 13 and 14
5. Calculate the updated values of pressure as $P^{n+1} = P^n + \alpha \delta P$, where α is a relaxation parameter
6. Use this new value of pressure as the new guess values of pressure and repeat steps 2 to 5 until convergence. Equation 16 is basically the continuity equation based on the starred values of the velocities. At convergence, the value of b should be zero.

The above discretized equations for the grid points are assembled in a matrix form and they are solved iteratively using different techniques. Suffice it to say that the direct solution of these matrices for practical problems with thousands of grid points is beyond the capability of most computers and the iterative techniques are dominant.

Of utmost importance is the implementation of boundary conditions. After all, it is the boundary conditions that drives the flow. In general, the velocities are zero on the surfaces due to the no slip condition. The inflow boundary is specified, if the incoming flow is known and the pressure at the outflow boundary is assumed to be zero. The grids are somehow distributed, so the grid points for the pressure do not lie on the surfaces. A detailed explanation of the boundary condition is beyond the scope of this article.

Computational Fluid Dynamics is a rich field with years of research and development spent on this topic. This article is only a high level overview of a special kind of numerical scheme in CFD. The intent is to familiarize readers with the very basics of CFD, so hopefully they can have a better appreciation of what is behind the commercial codes. To become an expert computational fluid dynamicist, requires taking multiple comprehensive courses in CFD and years of research.

References:

1. Patankar, S., "Numerical heat transfer and fluid flow", Hemisphere Publishing, ISBN: 0-07-048740-5